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Correspondence between the *XXZ* model in roots of unity and the one-dimensional quantum Ising chain with different boundary conditions

F C Alcaraz[1], **A A Belavin**[2] **and R A Usmanov**[2]

[1] Universidade Federal de São Carlos, 13565-905, São Carlos, SP Brazil
[2] Landau Institute for Theoretical Physics, Chernogolovka, Moscow region 142432, Russia

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Abstract

We consider the integrable *XXZ* model with special open boundary conditions that renders its Hamiltonian $SU(2)_q$ symmetric, and the one-dimensional quantum Ising model with four different boundary conditions. We show that for each boundary condition the Ising quantum chain is given exactly by the minimal model of integrable lattice theory $LM(3,4)$. This theory is obtained as the result of the quantum group reduction of the *XXZ* model at anisotropy $\Delta = (q+q^{-1})/2 = \sqrt{2}/2$, with a number of sites that depends on the type of imposed boundary condition.

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1. Introduction

Finite-width transfer matrices for four different boundary conditions on the cylinder were defined and studied in [1, 2], where it was shown that in each specific case, these matrices, which depend on a spectral parameter, form a commuting family and satisfy the same functional equation, which allows the evaluation of their eigenvalues.

As shown in [10], the logarithmic derivative of the above transfer matrices computed at the zero value of the spectral parameter is exactly the Hamiltonian of the one-dimensional quantum Ising chain with some boundary terms. The latter are different for each type of boundary condition of the two-dimensional model. Thus there is a direct connection between the critical Ising model on a two-dimensional lattice and the one-dimensional quantum Ising chain.

Based on numerical analysis on finite chains in [3, 7], it was realized that some of the eigenenergies of the *XXZ* Hamiltonian with the boundary condition that renders this Hamiltonian $SU(2)_q$ symmetric at the anisotropy $\Delta = (q+q^{-1})/2 = \sqrt{2}/2$ exactly coincide with some of the eigenenergies of the quantum Ising chain. Exact correspondences among the energies of both Hamiltonians was also observed in [11, 12], in the case of toroidal boundary conditions.

The notion of quantum group reduction of the integrable *XXZ* model with open boundary conditions (OBC) in roots of unity was introduced in [4]. The model given as a result of quantum group reduction is denoted by $LM(p,\ p+1)$ (p is a chain parameter) and called the minimal model of integrable lattice theory [5]. The thermodynamic limit ($N \to \infty$) of $LM(p,\ p+1)$ is $M(p,\ p+1)$, the ordinary minimal model of onformal field theory (CFT) with the Virasoro central charge $c = 1 - 6/p(p+1)$.

The integrability of the *XXZ* model with OBC manifests itself in the presence of a commuting family of transfer matrices found by Sklyanin [8, 9]. Sklyanin's transfer matrices of the *XXZ* model after quantum group reduction also satisfy some functional equations as shown in [5]. In the case of $p = 3$ these equations coincide with the functional equations for the Ising transfer matrices. This fact supports an equivalence of $LM(3, 4)$ and the Ising model.

In the present paper, we show that, in fact, these two models coincide exactly for all types of boundary conditions on the Ising lattice introduced in [2]. Namely, the following statements are true:

1. The *XXZ* chain with an odd number of sites $(2L+1)$ after the quantum group reduction of the configuration space is equivalent to the L-site Ising chain with mixed boundary conditions.
2. The configuration space of the $2L$-site *XXZ* chain after the quantum group reduction can be decomposed into a direct sum of two subspaces with the same dimension 2^{L-1}. These subspaces form two different representations of the Temperley–Lieb algebra $\mathcal{T}_{2L-1}$ and are eigensubspaces of the Casimir operator of $U_q(sl(2))$. We will denote by V_0 and V_1 the subspaces corresponding to the eigenvalues $(S^2)_q = \sqrt{2}$ and $(S^2)_q = 0$, respectively. Then the identification must be made as follows:
   (a) the whole complex of eigenvalues of the Hamiltonian of the $2L$-site *XXZ* chain computed on vectors from V_0 coincide with the spectrum of the $(L-1)$-site Ising chain with the boundary conditions (++) if L is even and the boundary conditions (+−) if L is odd;
   (b) the whole complex of eigenvalues of the Hamiltonian of the $2L$-site *XXZ* chain computed on vectors from V_1 coincide with the spectrum of the $(L-1)$-site Ising chain with the boundary conditions (++) if L is odd and the boundary conditions (+−) if L is even.
3. The spectrum of the L-site Ising chain with free boundary conditions coincide with the united spectrum of the $(L-1)$-sites Ising chains with the boundary conditions (++) and (+−).

The plan of the paper is as follows. In section 2 we define the Ising model with different cylindrical boundary conditions. In section 3, we introduce families of transfer matrices for four types of boundary conditions according to [2] and find the connection between these transfer matrices and Hamiltonians of the one-dimensional Ising chains. In section 4, we consider the integrable *XXZ* model with special open boundary conditions. In section 5, we then investigate two realizations of the Temperley–Lieb algebra in terms of dynamic variables of the Ising and *XXZ* chains. In section 6, we identify the Ising chain and $LM(3, 4)$. In section 7, we formulate some questions for the future. In appendices A and B we show some useful relations used in section 5.

2. The basic definitions

We consider two types of two-dimensional Ising lattices defined as follows [2]:

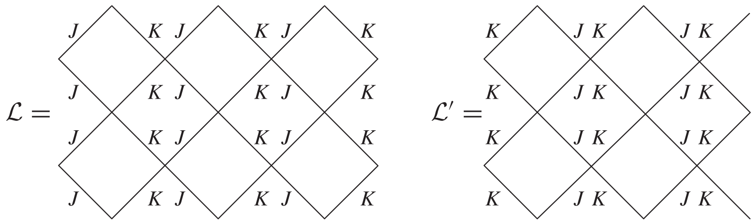

Figure 1. The square lattices $\mathcal{L}$ and $\mathcal{L}'$ where the Ising variables are attached. The links connecting variables with the coupling constants J and K are also shown.

1. a finite-width square lattice rotated by 45° with each row having L or $L-1$ faces;
2. a similar lattice except that each row has L faces.

Vertically, both lattices have columns of L' faces. We identify the first row of faces with the $(L'+1)$th row (the cylindrical boundary conditions). A lattice of the first type is denoted by the symbol $\mathcal{L}$ and a lattice of the second type by the symbol $\mathcal{L}'$ (see figure 1). Lattice $\mathcal{L}$ consists of $2L$ and $\mathcal{L}'$ of $2L+1$ zigzagging columns. Let us denote this number by N; i.e. $N = 2L$ for $\mathcal{L}$ and $N = 2L+1$ for $\mathcal{L}'$. Furthermore, we take L to be fixed.

On these lattices we can define the Ising model by attaching at each lattice site a spin variable taking the values +1 or −1. Since, in contrast with the toroidal model, the boundary spins can take arbitrary values we can consider distinct types of boundary conditions:

- ++: we choose the lattice $\mathcal{L}$ and fix the spins at the left and right boundaries to be +1;
- +−: we choose the lattice $\mathcal{L}$ and fix the left boundary spins to be + 1 and the right boundary spins to be −1;
- *mixed boundaries*: we choose the lattice $\mathcal{L}'$ and fix the left boundary spins to be +1, but place no restriction on the right boundary spins;
- *free boundaries*: we choose the lattice $\mathcal{L}$ with no restrictions on the boundary spins.

The lattices with the other possible boundary conditions, like, for example, −− or −+, are clearly related to the above ones.

3. Transfer matrices

The set of spins in some row that are not fixed by boundary conditions are denoted by Φ. We define transfer matrices as follows [2]:

1. *For free boundaries*:

$$T_{\Phi,\Phi'} = \sum_{\Phi''} \exp\left(J \sum_{j=1}^{L} \sigma''_j(\sigma_j + \sigma'_j) + K \sum_{j=1}^{L} \sigma''_{j+1}(\sigma_j + \sigma'_j) \right).$$

2. *For fixed boundary conditions*:

$$T_{\Phi,\Phi'} = \sum_{\Phi''} \exp\left(2J\sigma''_1 + K \sum_{j=1}^{L-1} \sigma''_j(\sigma_j + \sigma'_j) + J \sum_{j=1}^{L-1} \sigma''_{j+1}(\sigma_j + \sigma'_j) + 2\tau K \sigma''_L \right)$$

where $\tau = 1$ or $\tau = -1$, for the boundary conditions (++) or (+−) respectively.

3. *For mixed boundary conditions*:

$$T_{\Phi,\Phi'} = \sum_{\Phi''} \exp\left(2K\sigma_1'' + J\sum_{j=1}^{L}\sigma_j''(\sigma_j + \sigma_j') + K\sum_{j=1}^{L}\sigma_{j+1}''(\sigma_j + \sigma_j')\right).$$

Here, Φ, Φ'' and Φ' are three successive sets of spins, and we sum over Φ''. Transfer matrices are defined on the lattice $\mathcal{L}$ in the case of free and fixed boundary conditions and on the lattice $\mathcal{L}'$ in the case of mixed boundary conditions.

The transfer matrices defined above have a remarkable property:

$$T(J, K)T(J', K') = T(J', K')T(J, K)$$

(that is they commute with each other) if

$$\sinh(2K)\sinh(2J) = \sinh(2K')\sinh(2J'). \tag{1}$$

We consider the critical Ising model; therefore $\sinh(2J)\sinh(2K) = 1$. Following Baxter, we use the parametrization

$$\sinh(2J) = \cot(u) \qquad \sinh(2K) = \tan(u)$$

where $0 < u < \pi/2$ is a spectral parameter.

As shown in [2], the transfer matrices $T(u)$ satisfy the following functional equation:

$$T(2u)T(2u + \tfrac{1}{2}\pi) = \frac{\cos^{2(N+1)}(2u) - \sin^{2(N+1)}(2u)}{\cos(4u)} \frac{2^N(-1)^L}{(\sin(2u)\cos(2u))^N} \tag{2}$$

in the case of fixed boundary conditions and

$$T(2u)T(2u + \tfrac{1}{2}\pi) = \frac{\cos^{2(N+1)}(2u) - \sin^{2(N+1)}(2u)}{\cos(4u)} \frac{2^{N+2}(-1)^L}{(\sin(2u)\cos(2u))^N} \tag{3}$$

in the case of free and mixed boundary conditions, where N and L are introduced in section 2.

We now consider the Hamiltonian of the one-dimensional quantum Ising chain with L sites,

$$H^F_{\text{Ising}}(L) = \sum_{i=1}^{L-1}\sigma_j^z\sigma_{j+1}^z + \sum_{i=1}^{L}\sigma_j^x. \tag{4}$$

This Hamiltonian is related to the transfer matrix $T(u)$ of the Ising model with free boundary conditions. Apart from a harmless constant it is obtained by the operation $T^{-1}(0)\dot{T}(0)$, where $T^{-1}(u)$ is an inverse matrix and $\dot{T}(u)$ is the derivative of the matrix $T(u)$ with respect to its parameter.

For other boundary conditions the logarithmic derivative at $u = 0$ is slightly modified [10]:

1. for (++), we obtain the Hamiltonian of the $(L-1)$-site Ising chain
$$H^{++}_{\text{Ising}}(L) = H^F_{\text{Ising}}(L-1) + \sigma_1^z + \sigma_{L-1}^z$$
2. for (+−), we obtain the Hamiltonian of the $(L-1)$-site Ising chain
$$H^{+-}_{\text{Ising}}(L) = H^F_{\text{Ising}}(L-1) + \sigma_1^z - \sigma_{L-1}^z$$
3. for mixed boundary conditions, we obtain the Hamiltonian of the L-site Ising chain
$$H^M_{\text{Ising}}(L) = H^F_{\text{Ising}}(L) + \sigma_1^z$$
where H^F_{Ising} is the Hamiltonian of the model with free boundary conditions. The proof of this can be found in [10].

4. Minimal models of integrable lattice theory

We consider the one-dimensional *XXZ* chain with free boundary conditions [3]

$$\begin{aligned} H_{XXZ} &= \sum_{n=1}^{N-1}\left[\sigma_n^+\sigma_{n+1}^- + \sigma_n^-\sigma_{n+1}^+ + \frac{q+q^{-1}}{4}\sigma_n^z\sigma_{n+1}^z + \frac{q-q^{-1}}{4}(\sigma_n^z - \sigma_{n+1}^z)\right] \\ \sigma_n^{\pm} &= 1\otimes\cdots\otimes\sigma^{\pm}\otimes\cdots\otimes 1 \\ \sigma_n^z &= 1\otimes\cdots\otimes\sigma^{z}\otimes\cdots\otimes 1 \\ \sigma^+ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{5}$$

In the thermodynamic limit ($N \to \infty$) this Hamiltonian is gapless for $-1 \leqslant \Delta = (q+q^{-1})/2 \leqslant 1$. The model has remarkable properties. Besides the $U(1)$ symmetry, translated into its commutation with the z-component of the total magnetization $S^z = \frac{1}{2}\sum_{i=1}^{N}\sigma_i^z$, it was also shown in [4] that the Hamiltonian H_{XXZ} commutes with the quantum group $U_q(sl(2))$ including its generators X, Y and H, which are defined as

$$\begin{aligned} X &= \sum_{n=1}^{N} q^{\frac{1}{2}(\sigma_1^z+\cdots+\sigma_{n-1}^z)}\sigma_n^+ q^{-\frac{1}{2}(\sigma_{n+1}^z+\cdots+\sigma_N^z)} \\ Y &= \sum_{n=1}^{N} q^{\frac{1}{2}(\sigma_1^z+\cdots+\sigma_{n-1}^z)}\sigma_n^- q^{-\frac{1}{2}(\sigma_{n+1}^z+\cdots+\sigma_N^z)} \\ H &= \sum_{n=1}^{N}\frac{\sigma_n^z}{2} \end{aligned}$$

and satisfies the relations

$$[H, X] = X \qquad [H, Y] = -Y \qquad [X, Y] = \frac{q^{2H} - q^{-2H}}{q - q^{-1}}. \tag{6}$$

Furthermore, the densities

$$H_n = \sigma_n^+\sigma_{n+1}^- + \sigma_n^-\sigma_{n+1}^+ + \frac{q+q^{-1}}{4}\sigma_n^z\sigma_{n+1}^z + \frac{q-q^{-1}}{4}(\sigma_n^z - \sigma_{n+1}^z)$$

where $n = 1, \ldots, N-1$, also commute with the quantum group $U_q(sl(2))$.

Because of the quantum group symmetry the spectrum of the Hamiltonian can be classified according to the representation theory of the algebra $U_q(sl(2))$.

The representation theory of $U_q(sl(2))$ has been studied in detail in the case where q is not a root of unity. In this case, its representations are equivalent to those of the usual $U(sl(2))$, that is, the configuration space $(\mathbb{C}^2)^N$ of the spin chain can be split into a direct sum of irreducible highest-weight representations ρ_j (j is the highest weight), which are in one-to-one correspondence with the ordinary $sl(2)$ representations. For example, in the case $N = 4$, where $(\mathbb{C}^2)^4$ can be decomposed into $\rho_2 + 3\rho_1 + 2\rho_0$.

We study the case where $q^{p+1} = -1$ [4, 6]. In this case, the generators X and Y are nilpotent:

$$X^{p+1} = 0 \qquad Y^{p+1} = 0. \tag{7}$$

We consequently obtain a very different picture of the representations.

For example, if $q^4 = -1$ and $N = 4$ then $X^4 = 0$ and $Y^4 = 0$. The space $(\mathbb{C}^2)^4$ now decomposes into the sum of one 'bad' eight-dimensional representation (ρ_2, ρ_1) of type I and four other 'good' representations $2\rho_1 + 2\rho_0$ of type II [4]. There is an isomorphism of the type-II representations and the ordinary $U(sl(2))$ ones. The type-I representation can be considered as the result of gluing two representations ρ_2 and ρ_1. It is indecomposable but is not irreducible (it contains a three-dimensional invariant subspace).

In the general case with $q^{p+1} = -1$ [4, 6] the configuration space splits into the sum of 'bad' type-I representations with the highest weights $S_z \geqslant p/2$ and 'good' type II representations with highest weights $S_z < p/2$ which are simultaneously not subspaces of some 'bad' ones. The highest-weight vectors v_j of the good representations can be characterized by the condition

$$v_j \in V_p \equiv \operatorname{Ker} X / \operatorname{Im} X^p. \tag{8}$$

Because $U_q(sl(2))$ commutes with H_{XXZ}, we can normally restrict its action on the space V_p. The model resulting from this quantum group reduction model is called $LM(p,\ p + 1)$ [5].

The number of representations with the highest weight $j < p/2$ in the decomposition of $(\mathbb{C}^2)^N$ equals the number of restricted paths of length N beginning at zero and ending at j. The restriction means that a path cannot cross the straight lines $j = 0$ and $p/2$. For example, if $p = 3$ and N is odd, then only the paths ending at $j = \frac{1}{2}$ are permissible, and the number of paths is therefore 2^{N-1}.

A so-called Sklyanin transfer matrix [8, 9]

$$T_{1/2}(u) = (-1)^N \operatorname{tr}(\mathrm{e}^{-\sigma^z(u+\eta)} L(u)\, \mathrm{e}^{\sigma^z u} L^{t\otimes t}(u))$$

is related to Hamiltonians of the *XXZ* chain. Here $L(u)$ is a monodromy matrix

$$L(u) = R_N(u) \dots R_1(u)$$

and $R(u)$ is given by the expression

$$\begin{pmatrix} \begin{pmatrix} \sin(u+\eta) & 0 \\ 0 & \sin(u) \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ \sin(\eta) & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & \sin(\eta) \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} \sin(u) & 0 \\ 0 & \sin(u+\eta) \end{pmatrix} \end{pmatrix}.$$

The relation

$$H_{XXZ} = \frac{\sin(\eta)}{2} \frac{\mathrm{d} \log T_{1/2}(u)}{\mathrm{d}u}\bigg|_{u=0} + \frac{\sin^2(\eta)}{2\cos(\eta)} - \frac{N}{2}\cos(\eta)$$

holds. The transfer matrices commute with each other under different values of the spectral parameter and therefore with the Hamiltonian

$$[T_{1/2}(u), T_{1/2}(v)] = 0 \qquad [T_{1/2}(u), H_{XXZ}] = 0.$$

As shown in [5], after the quantum group reduction, $T_{1/2}(u)$ satisfies the functional equation

$$T_{1/2}(u) T_{1/2}\left(u + \frac{\pi}{4}\right) = 2^{-2N+1} \frac{\cos^{2(N+1)}(2u) - \sin^{2(N+1)}(2u)}{\cos(4u)}. \tag{9}$$

Comparing equations (2) and (9), we can see that in the case of even N and fixed boundary conditions, the matrices $T_{1/2}(u)$ and $2^{1/2-2N}(\sin(4u))^L T_{\mathrm{Ising}}(2u)$ satisfy the same functional equation. (Here, the transfer matrix of the Ising model with fixed boundary conditions is denoted by $T_{\mathrm{Ising}}(u)$ rather than $T(u)$ as it was in (2).) Their eigenvalues hence coincide, and the two matrices are therefore equivalent.

In the case of even N and free boundary conditions, the matrices $T_{1/2}(u)$ and $2^{-1/2-2N}(\sin(4u))^L T_{\mathrm{Ising}}(2u)$ also satisfy the same functional equation. Here, $T_{\mathrm{Ising}}(u)$ is the transfer matrix of the Ising model with free boundary equations.

Similarly, in the case of odd N, that is, for mixed boundary conditions, the matrices $T_{1/2}(u)$ and $2^{1/2-2N}(\sin(2u))^{L+1}(\cos(2u))^L T_{\mathrm{Ising}}(2u)$ satisfy the same functional equation. Here, $T_{\mathrm{Ising}}(u)$ is the transfer matrix of the Ising model with mixed boundary conditions.

Once again, we emphasize that while in the Ising model we have L spins in the related *XXZ* chain we have $N = 2L$ spins if the boundary conditions in the Ising model are free or fixed, and $N = 2L + 1$ in the case where the Ising chain has mixed boundary conditions. This means that the dimension of $T_{1/2}$, which is 2^N is always bigger than the corresponding dimension 2^L of the related T_{Ising}.

That the transfer matrices in the two models satisfy functional equations of similar form suggests the essential identity of the models. In what follows, we establish this identity of the *XXZ* and Ising models and show how it can be realized.

5. Temperley–Lieb algebra

We recall the definition of the Temperley–Lieb algebra as an algebra with the generators $e_i,\ i = 1, \ldots, n$, satisfying the relations

$$e_i e_{i\pm1} e_i = e_i \qquad e_i^2 = (q + q^{-1})e_i \qquad [e_i, e_j] = 0 \qquad |i - j| > 1. \tag{10}$$

The Temperley–Lieb algebra with n generators is denoted by $\mathcal{T}_n$. Furthermore, we assume that $q = \mathrm{e}^{\mathrm{i}\pi/4}$.

There are realizations of the Temperley–Lieb algebra in terms of the dynamic variables of Ising and *XXZ* model. Namely, it can be verified that:

1. the expressions

$$e_{2i-1} = \frac{1}{\sqrt{2}}(\sigma_i^z \sigma_{i+1}^z + 1) \qquad i = 1, \ldots, L$$

$$e_{2i} = \frac{1}{\sqrt{2}}(\sigma_i^x + 1) \qquad i = 1, \ldots, L - 1$$

give a realization of $\mathcal{T}_{2L-1}$;

2. the expressions

$$e_1 = \frac{1}{\sqrt{2}}(\sigma_1^z + 1)$$

$$e_{2i} = \frac{1}{\sqrt{2}}(\sigma_i^z \sigma_{i+1}^z + 1) \qquad i = 1, \ldots, L$$

$$e_{2i-1} = \frac{1}{\sqrt{2}}(\sigma_i^x + 1) \qquad i = 2, \ldots, L$$

give a realization of $\mathcal{T}_{2L}$;

3. the expressions

$$e_1 = \frac{1}{\sqrt{2}}(\sigma_1^z + 1)$$

$$e_{2i-1} = \frac{1}{\sqrt{2}}(\sigma_i^z \sigma_{i+1}^z + 1) \qquad i = 2, \ldots, L$$

$$e_{2i} = \frac{1}{\sqrt{2}}(\sigma_i^x + 1) \qquad i = 1, \ldots, L-1$$

$$e_{2L-1} = \frac{1}{\sqrt{2}}(\sigma_{L-1}^z + 1)$$

   give a realization of $\mathcal{T}_{2L-1}$;
4. the expressions

$$e_1 = \frac{1}{\sqrt{2}}(\sigma_1^z + 1)$$

$$e_{2i-1} = \frac{1}{\sqrt{2}}(\sigma_i^z \sigma_{i+1}^z + 1) \qquad i = 2, \ldots, L$$

$$e_{2i} = \frac{1}{\sqrt{2}}(\sigma_i^x + 1) \qquad i = 1, \ldots, L-1$$

$$e_{2L-1} = \frac{1}{\sqrt{2}}(-\sigma_{L-1}^z + 1)$$

   give a realization of $\mathcal{T}_{2L-1}$.

Using these expressions, we see that the Hamiltonian of the Ising model for all four boundary conditions is

$$H_{\text{Ising}} = \sum_{i=1}^{N-1} \left(e_i\sqrt{2} - 1\right)$$

where $N = 2L$ for free and fixed boundary conditions and $N = 2L + 1$ for mixed boundary conditions.

In terms of the dynamic variables of the *XXZ* model, the expressions

$$e_i = -H_i + \frac{\sqrt{2}}{4}$$

where $i = 1, \ldots, N-1$, give a realization of $\mathcal{T}_{N-1}$, and the Hamiltonian of the N-site *XXZ* chain is

$$H_{XXZ} = -\sum_{i=1}^{N-1}\left(e_i - \frac{\sqrt{2}}{4}\right).$$

That the Hamiltonians of *XXZ* and Ising models have the same form in terms of generators of the Temperley–Lieb algebra supports the equivalence of the two models.

6. Identification of $LM(3, 4)$ and the Ising model

6.1. The XXZ chain with an odd number of sites and the Ising chain with mixed boundary conditions

We consider the $(2L + 1)$-site *XXZ* chain and the L-site Ising chain with mixed boundary conditions. The same algebra $\mathcal{T}_{2L}$ corresponds to each of them.

The densities H_i of the Hamiltonian of the *XXZ* model commute with the quantum group $U_q(sl(2))$. Therefore, the configuration space of the *XXZ* model after the quantum group reduction forms a representation of the algebra $\mathcal{T}_{2L}$ as it was before the reduction. This representation, whose vectors are in one-to-one correspondence with restricted paths (just such as in the RSOS model) of length $2L + 1$ and height $2j = 1$, has the dimension 2^L and is irreducible. The realization of the Temperley–Lieb algebra on the Ising configuration space gives the same representation. The equivalence of the two models is implied these facts and the fact that their Hamiltonians have the same form in terms of the generators of the Temperley–Lieb algebra.

This statement can be check numerically by comparing the eigenvalues of the operator

$$\sum_{i=1}^{N-1} e_i$$

computed on vectors from the configuration space of $LM(3, 4)$ and on vectors from the configuration space of the Ising model for small L.

6.2. The XXZ chain with an even number of sites and the Ising chain with fixed boundary conditions

We now consider the *XXZ* chain with $N = 2L$ sites and the $(L - 1)$-site Ising chain with fixed boundary conditions. In the previous section, it was shown that the same algebra $\mathcal{T}_{N-1}$ corresponds to each of them.

The dimension of the configuration space of each of the two Ising Hamiltonians in this case is half the dimension of the configuration space of $LM(3, 4)$ V_3. The space V_3 has the dimension 2^L and is decomposed into the sum of two subspaces. Each of them is an eigensubspace of the Casimir operator

$$(S^2)_q = YX + \left(\frac{q^{H+1/2} - q^{-H-1/2}}{q - q^{-1}}\right)^2 - \left(\frac{q^{1/2} - q^{-1/2}}{q - q^{-1}}\right)^2$$

one corresponding to the eigenvalue $(S^2)_q = 0$, and the other corresponding to the eigenvalue $(S^2)_q = \sqrt{2}$. The first subspace is denoted by V_0 and the second by V_1. Because the Casimir operator commutes with the algebra $\mathcal{T}_{2L-1}$, V_0 and V_1 form a representation of that algebra. The dimension of each subspace is 2^{L-1}, which is exactly the dimension of the configuration space of each of the two Ising chains.

As shown in appendix A,

$$2^{2N} q^{-N} \mathrm{e}^{-2\mathrm{i}Nu} T_{1/2}(u \to -\mathrm{i}\infty) = \left\{(q - q^{-1})^2 (S^2)_q + (q + q^{-1})\right\}$$

and since the subspaces V_0 and V_1 are eigensubspaces of the matrix on the right-hand side, we can hence see that V_0 and V_1 are simultaneously eigensubspaces of the matrix on the left-hand side, with eigenvalues -1 and $+1$, respectively.

The densities H_i of the Hamiltonian of the *XXZ* model commute with the quantum group $U_q(sl(2))$. Therefore, the configuration space of the *XXZ* model after the quantum group reduction forms a representation of the algebra $\mathcal{T}_{2L}$ as it was before the reduction. This representation, whose vectors are in one-to-one correspondence with restricted paths of length $2L$ and height $2j = 0$ and $2j = 2$, has the dimension 2^L and can be decomposed into the direct sum of two irreducible representations corresponding to two different heights of paths. The dimension of each representation equals 2^{L-1}. The equivalence of the two models in each of the cases in question is implied by these facts and the fact that their Hamiltonians have the same form in terms of the generators of the Temperley–Lieb algebra.

It was shown in [2] that $T_{\text{Ising}}(-\mathrm{i}\infty) = \pm 2^L$, where the upper sign corresponds to the boundary conditions (++) and the lower sign corresponds to the boundary conditions (+−). The matrices $2^{1-2N}(\sin(4u))^L T_{\text{Ising}}(2u)$ and $T_{1/2}(u)$ become equivalent after the quantum group reduction. We can therefore write

$$T_{1/2}(u) = 2^{1-2N}(\sin(4u))^L T_{\text{Ising}}(2u)$$

in the $u \to -\mathrm{i}\infty$ limit. Substituting $q = \mathrm{e}^{\mathrm{i}\pi/4}$, we obtain

$$\frac{1}{2^L} T_{\text{Ising}}(-\mathrm{i}\infty) = (-1)^L \big\{ -\sqrt{2}(S^2)_q + 1 \big\}. \tag{11}$$

Because the eigenvalues of the Casimir operator equal 0 and $\sqrt{2}$, we can conclude that for even L, the configuration space of the Ising chain with the boundary conditions (++) corresponds to V_0 and the configuration space of the Ising chain with the boundary conditions (+−) corresponds to V_1. For odd L, the boundary conditions (++) correspond to V_1 and the boundary conditions (+−) correspond to V_0.

6.3. The XXZ chain with an even number of sites and the Ising chain with free boundary conditions

We now consider the *XXZ* chain with $N = 2L$ sites and the L-site Ising chain with free boundary conditions. The algebra $\mathcal{T}_{2L-1}$ corresponds to each of them. The dimensions of the configuration spaces of the Ising chain and $LM(3,4)$ both equal 2^L.

In the case of free boundary conditions the Ising quantum chain has a $Z(2)$ symmetry translated by the commutation of H^F_{Ising} with the parity operator C given by

$$C = \sigma_1^x \sigma_2^x \cdots \sigma_L^x.$$

Hence, the configuration space of the Ising chain with free boundary conditions is decomposed into the sum of two sectors, corresponding to two eigenvalues of C. These two sectors are denoted by C_+ and C_-.

We show in appendix B that C is related to the limit of the transfer matrix of the L-site Ising model with free boundary conditions by

$$C = \frac{1}{2^{L+1}} T_{\text{Ising}}(-\mathrm{i}\infty).$$

As with fixed boundary conditions, we can obtain the identity

$$C = \frac{1}{2^{L+1}} T_{\text{Ising}}(-\mathrm{i}\infty) = (-1)^L \big\{ -\sqrt{2}(S^2)_q + 1 \big\} \tag{12}$$

from which we can see that there is a one-to-one correspondence between the sectors in the configuration space of the Ising chain and the subspaces V_0 and V_1. Namely, if L is even, then the sector C_+ corresponds to V_0 and the sector C_- to V_1. If L is odd, then C_- corresponds to V_0 and C_+ to V_1.

We have thus proved all statements formulated in the introduction.

7. Some questions for future work

We now formulate some questions for future work, which we will try to answer in one of our subsequent papers.

In the previous sections we have considered the critical Ising model and the homogeneous *XXZ* model and proved their equivalence. It seems that our proof can be generalized to the case of the non-critical inhomogeneous Ising model given by the Hamiltonian

$$H_{\text{Ising}}(L) = \sum_{i=1}^{L-1} a_i \sigma_i^z \sigma_{i+1}^z + \sum_{i=1}^{L} b_i \sigma_i^x$$

and the inhomogeneous *XXZ* model given by

$$H_{XXZ}(N) = \sum_{i=1}^{N-1} c_i H_i$$

where a_i, b_i and c_i are arbitrary coefficients.

The inhomogeneous Hamiltonian *XXZ* commutes with the quantum group $U_q(sl(2))$, since the densities H_i, $i = 1, \ldots, N-1$ themselves commute with it.

Furthermore, we can obtain the Hamiltonian *XXZ* by evaluating the logarithmic derivative of Sklyanin's transfer matrix but now we must use its inhomogeneous version, that is, the monodromy matrix L equals the product of R-matrices (as it was in section 4) evaluated at different values of the parameter u:

$$L(u) = R_N(u - u_N) \cdots R_2(u - u_2) R_1(u - u_1).$$

It can be found that

$$c_i = \frac{\sin\eta}{\sin^2\eta - \sin^2 u_i}$$

where $i = 1, \ldots, N-1$.

In particular, in the case

$$a_i = -1 \qquad b_i = -\lambda \qquad c_i = \frac{\lambda + 1}{2} + (-1)^i \frac{\lambda - 1}{2}$$

both Hamiltonians have the same form in terms of the generators of the Temperley–Lieb algebra:

$$H_{\text{Ising}}(L, \lambda) = -\sum_{i=1}^{N-1} \left(\frac{\lambda + 1}{2} + (-1)^i \frac{\lambda - 1}{2} \right) \left(e_i \sqrt{2} - 1 \right)$$

$$H_{XXZ}(N, \lambda) = -\sum_{i=1}^{N-1} \left(\frac{\lambda + 1}{2} + (-1)^i \frac{\lambda - 1}{2} \right) \left(e_i - \frac{\sqrt{2}}{4} \right)$$

with the generators of the Temperley–Lieb algebra in terms of the dynamic variables of the Ising model given by the old expressions and in terms of the dynamic variables of the *XXZ* model given by

$$e_i = -\frac{H_i}{(\lambda+1)/2 + (-1)^i(\lambda-1)/2} + \frac{q+q^{-1}}{4} \qquad i = 1, \ldots, N-1.$$

Consequently, the Hamiltonians of the *XXZ* and the Ising models, properly normalized

$$\tilde{H}_{\mathrm{Ising}}(L, \lambda) = H_{\mathrm{Ising}}(L, \lambda)/\sqrt{2} - C_N\sqrt{2}/2 \tag{13}$$

and

$$\tilde{H}_{XXZ}(N, \lambda) = H_{XXZ}(N, \lambda) - C_N\sqrt{2}/4 \tag{14}$$

with

$$C_N = \sum_{i=1}^{N}\left(\frac{\lambda+1}{2} + (-1)^i\frac{\lambda-1}{2}\right)$$

are expressed in a similar form in terms of the generators of the Temperley–Lieb algebra. This fact supports equivalence of the two models.

To illustrate this we exhibit in table 1 the eigenenergies of the *XXZ* chain with $N = 5$ sites and those of the $L = 2$ Ising chain with mixed boundary condition, both models at $\lambda = 2$. The eigenenergies of the *XXZ* chain are separated according to their z-magnetization S^z, and the spin S of the highest weights are also shown. The energy levels forming inhomogeneous $LM(3, 4)$ have a superscript (+), and we see their equality with the energy levels of the related Ising quantum chain. The level with an asterisk symbol, although having spin $S = \frac{1}{2}$ do not belong to inhomogeneous $LM(3, 4)$ since it is degenerated with another level with $S = \frac{5}{2}$.

Table 1. Eigenenergies of the normalized Hamiltonians $\tilde{H}_{XXZ}$ and $\tilde{H}^M_{\mathrm{Ising}}$ given by (13) and (14). The eigenenergies of the *XXZ* Hamiltonian are separated into the sectors labelled by the z-magnetization S^z. The spins S of the highest weights are also shown. The levels marked by (+) form $LM(3, 4)$ and coincide with those of the Ising quantum chain. The level marked by (∗) does not belong to $LM(3, 4)$ since it is degenerated with another level with $S = \frac{5}{2}$.

	$\tilde{H}_{XXZ}(N = 5, \lambda = 2)$			$\tilde{H}^M_{\mathrm{Ising}}(L = 2, \lambda = 2)$
S	$S^z = \pm\frac{1}{2}$	$S^z = \pm\frac{3}{2}$	$S^z = \pm\frac{5}{2}$	
$\frac{1}{2}$	$-7.338\,21^{(+)}$			$-7.338\,21$
$\frac{1}{2}$	$-4.888\,72^{(+)}$			$-4.888\,72$
$\frac{3}{2}$	$-4.516\,33$	$-4.516\,33$		
$\frac{1}{2}$	$-3.596\,56^{(+)}$			$-3.596\,56$
$\frac{3}{2}$	$-3.245\,57$	$-3.245\,57$		
$\frac{1}{2}$	$-1.147\,07^{(+)}$			$-1.147\,07$
$\frac{3}{2}$	$-0.997\,97$	$-0.997\,97$		
$\frac{5}{2}$	0	0	0	
$\frac{1}{2}$	0 (*)			
$\frac{3}{2}$	0.273 69	0.273 69		

Table 2. Part of the eigenenergies of the normalized Hamiltonians $\tilde{H}_{XXZ}$ given by (14) and the corresponding ones of the $\tilde{H}^{++}_{\text{Ising}}$ and $\tilde{H}^{+-}_{\text{Ising}}$ given by (13). The eigenenergies of the *XXZ* Hamiltonian are separated into the sectors labelled by the z-magnetization S^z. The spins S of the highest weights are also shown. The levels with the superscript (0) and (1) belongs to the sectors V_0 and V_1 of $LM(3,4)$, respectively. The levels marked by $(*)$ does not belong to $LM(3,4)$ since they are degenerated with others levels with $S=2$.

	$\tilde{H}_{XXZ}(N=6,\lambda=2)$			$\tilde{H}^{++}_{\text{Ising}}(L=2,\lambda=2)$	$\tilde{H}^{+-}_{\text{sing}}(L=2,\lambda=2)$
S	$S^z=0$	$S^z=\pm1$	$S^z=\pm2$		
0	$-8.348\,40^{(0)}$			$-8.348\,40$	
1	$-8.074\,13^{(1)}$	$-8.074\,13^{(1)}$			$-8.074\,13$
1	$-5.656\,85^{(1)}$	$-5.656\,85^{(1)}$			$-5.656\,85$
0	$-5.382\,59^{(0)}$			$-5.382\,59$	
2	$-4.670\,83$	$-4.670\,83$	$-4.670\,83$		
1	$-4.670\,83(*)$	$-4.670\,83(*)$			
1	$-4.516\,91^{(1)}$	$-4.516\,91^{(1)}$			$-4.516\,91$
0	$-4.242\,64^{(0)}$			$-4.242\,64$	
2	$-3.702\,46$	$-3.702\,46$	$-3.702\,46$		
1	$-3.702\,46(*)$	$-3.702\,46(*)$			
0	$-1.823\,56^{(0)}$			$-1.823\,56$	
1	$-1.551\,09^{(1)}$	$-1.551\,09^{(1)}$			$-1.551\,09$

Table 3. Eigenenergies of the normalized Hamiltonians $\tilde{H}_{XXZ}$ and $\tilde{H}^{F}_{\text{Ising}}$ given by (13) and (14), respectively. The energies of the *XXZ* Hamiltonian are separated into the sectors labelled by the z-magnetization S^z, and those of the Ising chain are separated according to their parity $C=\pm1$. The spins S of the highest weights of the *XXZ* Hamiltonian are also shown. The levels with the superscript (0) and (1) belongs to the sectors V_0 and V_1 of $LM(3,4)$, respectively. The level marked by $(*)$ does not belong to $LM(3,4)$ since it is degenerated with another level with $S=2$.

	$\tilde{H}_{XXZ}(N=4,\lambda=2)$			$\tilde{H}^{F}_{\text{Ising}}(L=2,\lambda=2)$	
S	$S^z=0$	$S^z=\pm1$	$S^z=\pm2$	$C=+1$	$C=-1$
0	$-6.451\,0^{(0)}$			$-6.451\,01$	
1	$-4.242\,64^{(1)}$	$-4.242\,64^{(1)}$			$-4.242\,64$
1	$-2.828\,43^{(1)}$	$-2.828\,43^{(1)}$			$-2.828\,43$
0	$-0.620\,06^{(0)}$			$-0.620\,06$	
1	0(*)	0(*)			
2	0	0	0		

In table 2 we illustrate relations between the two models by showing the eigenspectra of $\tilde{H}^{++}_{\text{Ising}}(L=2,\lambda=2)$, $\tilde{H}^{+-}_{\text{Ising}}(L=2,\lambda=2)$ and part of the eigenspectrum of $\tilde{H}_{XXZ}(N=6,\lambda=2)$. The eigenenergies of the *XXZ* chain are separated in the S^z sectors and the spin S of the highest weights are also shown. The energy levels forming V_0 ($S=0$) and V_1 ($S=1$) have a superscript (0) and (1), respectively. The eigenenergies having an asterisk does not belong to V_1 since they are degenerated with the other eigenenergies with $S>1$. We see from this table that inhomogeneous $LM(3,4)$ is obtained by gluing together the eigenspectra of the related Ising chains.

In table 3 we illustrate correspondences between the Ising model and inhomogeneous $LM(3,4)$ by showing the eigenenergies of $\tilde{H}_{XXZ}(N=4,\lambda=2)$ and $\tilde{H}^{F}_{\text{Ising}}(L=2,\lambda=2)$. The energies of the *XXZ* chain are separated according to the S^z sector and those of the Ising chain are separated according to their parity $C=\pm1$. The corresponding spin S of the levels are shown and the energies forming V_0 and V_1 have a superscript 0 and 1, respectively. The energy level with an asterisk does not belong to V_1 since it degenerates with another energy with $S = 2$. From this table the exact correspondence between inhomogeneous $LM(3, 4)$ and the Ising chain is clear.

Thus the relations between the two models are very similar to those for the critical homogeneous models.

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Appendix A. The limit of Sklyanin's transfer matrix

The limit of the monodromy matrix $L(u)$ as $u \to -\mathrm{i}\infty$ equals

$$L(u \to -\mathrm{i}\infty) = (-\mathrm{i})^N 2^{-N} \mathrm{e}^{\mathrm{i}(u+\eta/2)(N-1)} \begin{pmatrix} \mathrm{e}^{\mathrm{i}(u+\eta/2)} q^H & (q - q^{-1}) X_0 \\ (q - q^{-1}) X & \mathrm{e}^{\mathrm{i}(u+\eta/2)} q^{-H} \end{pmatrix}$$

where

$$\begin{aligned} H &= \sum_{n=1}^{N} \frac{\sigma_n^z}{2} \qquad q = \mathrm{e}^{\mathrm{i}\eta} \\ X_0 &= \sum_{n=1}^{N} q^{-\frac{1}{2}(\sigma_1^z + \cdots + \sigma_{n-1}^z)} \sigma_n^- q^{-\frac{1}{2}(\sigma_{n+1}^z + \cdots + \sigma_N^z)}. \end{aligned} \tag{A1}$$

We can write our transfer matrix as

$$T_{1/2}(u) = (-1)^N \sum_{m,n=1}^{2} q^{2m-3} \mathrm{e}^{2\mathrm{i}(m-n)u} L_{m,n}(u) L_{m,n}^t(u) \tag{A2}$$

Where the indices (m, n) are the indices of the elements of $L(u)$ and $L_{m,n}^t$ is an operator in 'quantum' space $(\mathbb{C}^2)^N$ given by transposition of $L_{m,n}$. We find the limit $T(u \to -\mathrm{i}\infty)$. Using (A2), we obtain

$$\begin{aligned} T_{1/2}(u \to -\mathrm{i}\infty) &= (-1)^N 2^{-2N} q^{N-1} \mathrm{e}^{2\mathrm{i}u(N-1)} \\ &\quad \times \big(q^{-1} L_{1,1} L_{1,1}^t + q L_{2,2} L_{2,2}^t + q \mathrm{e}^{2\mathrm{i}u} L_{2,1} L_{2,1}^t q^{-1} \mathrm{e}^{-2\mathrm{i}u} L_{1,2} L_{1,2}^t\big) \\ &= 2^{-2N} q^{N-1} \mathrm{e}^{2\mathrm{i}u(N-1)} \big(q^{-1} \mathrm{e}^{2\mathrm{i}u} q q^{2H} + q \mathrm{e}^{2\mathrm{i}u} (q - q^{-1})^2 X X^t \\ &\quad + q^{-1} \mathrm{e}^{-2\mathrm{i}u} (q - q^{-1})^2 X_0 X_0^t + q \mathrm{e}^{2\mathrm{i}u} q q^{-2H}\big). \end{aligned} \tag{A3}$$

The last term vanishes exponentially, and we can therefore ignore it. Using $X^t = Y$ (see the previous section), we obtain

$$T_{1/2}(u \to -\mathrm{i}\infty) = 2^{-2N} q^N \mathrm{e}^{2\mathrm{i}Nu} \big\{q^{-1} q^{2H} + q q^{-2H} + (q - q^{-1})^2 X Y\big\}. \tag{A4}$$

Using

$$XY = YX + \frac{q^{2H} - q^{-2H}}{q - q^{-1}}$$

we obtain

$$T_{1/2}(u \to -\mathrm{i}\infty) = 2^{-2N} q^N \mathrm{e}^{2\mathrm{i}Nu} \left\{ (q - q^{-1})^2 (S^2)_q + (q + q^{-1}) \right\} \tag{A5}$$

where $(S^2)_q$ is Casimir of the algebra $U_q(sl(2))$:

$$(S^2)_q = YX + \left(\frac{q^{H+1/2} - q^{-H-1/2}}{q - q^{-1}} \right)^2 - \left(\frac{q^{1/2} - q^{-1/2}}{q - q^{-1}} \right)^2.$$

Appendix B. The operator C and the limit of the transfer matrix of the Ising model with free boundary conditions

We prove that

$$C = \frac{1}{2^{L+1}} T_{\mathrm{Ising}}(-\mathrm{i}\infty)$$

where $T_{\mathrm{Ising}}(u)$ is the transfer matrix of the Ising model.

In the case with free boundary conditions, an element of the transfer matrix equals

$$\sum_{\sigma''} \exp\left[J\sigma_1''(\sigma_1 + \sigma_1') \right] \prod_{j=1}^{L} \exp\left[K\sigma_j''(\sigma_j + \sigma_j') + J\sigma_{j+1}''(\sigma_j + \sigma_j') \right] \exp\left[K\sigma_{L+1}''(\sigma_L + \sigma_L') \right]$$

$$= 2^{L+1} \prod_{j=1}^{L} \cosh[J(\sigma_1 + \sigma_1')] \cosh[K(\sigma_j + \sigma_j') + J(\sigma_j + \sigma_j')] \cosh[K(\sigma_L + \sigma_L')].$$

For $u = -\mathrm{i}\infty$,

$$\cosh(2J) = 0 \qquad \cosh(2K) = 0.$$

Hence, if there is even one pair σ_j, σ_j' such that $\sigma_j = \sigma_j'$, then the matrix element equals zero. We can see from this that $T(-\mathrm{i}\infty)$ is actually proportional to the product of all σ_i^x (with the coefficient of proportionality equalling 2^{L+1}), as this product has only those elements non-zero for which $\sigma_i = -\sigma_i'$ for all i. Hence we obtain the desired result.

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